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ANTIPLANE PERIODIC CONTACT PROBLEMS FOR A LAYER NON-UNIFORM ALONG ITS THICKNESS†

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Antiplane periodic contact problems for an elastic layer with a shear modulus which varies exponentially along its thickness are considered. The problems are reduced to an integral equation of the first kind with an irregular, periodic, difference kernel. A method which has been described previously [1, 2] is used for the approximate solution of this equation.

Similar problems for a uniform elastic layer were considered by another method in [3].

1. FORMULATION OF THE PROBLEMS

Suppose an elastic layer ($|x| < \infty$, $|x| < \infty$, $|y| \le h$) is rigidly fastened onto a base $y = 0$. The shear modulus of the layer varies along its depth as given by the relation

$$G(y) = G_0 e^{2\kappa y} \tag{1.1}$$

A periodic system of strip punches, to which a linear shear force T is applied, is arranged along the upper edge of the layer $y = h$. Antiplane deformation of the layer occurs under the action of these punches. The punches are arranged with a period of $2b$ and the area of contact of each punch with the surface of the layer has a length $2a$ ($a < b$). The punches are rigidly joined to the surface of the layer over their contact areas.

We shall consider two problems: (1) the punches are successively shifted by the forces T in different directions by an amount ε, (2) the punches are shifted in one direction by an amount ε. We shall call the punch which is symmetrically arranged with respect to the system of coordinates a primary punch. The directions of the forces T in the case of Problem 1 are shown in Fig. 1. In the case of Problem 2, all of the forces are directed in the same direction as the force which is applied to the primary punch.

We will now introduce notation and initial formulae. The components of the displacement vector are denoted by u, υ and w, where

$$u = \upsilon = 0, \quad w = w(x, y) \tag{1.2}$$

the shear strains

$$\gamma_{xz} = \partial w/\partial x, \quad \gamma_{yz} = \partial w/\partial y \tag{1.3}$$

and the shear stresses

$$\tau_{xz} = G(y)\gamma_{xz}, \quad \tau_{yz} = G(y)\gamma_{yz} \tag{1.4}$$

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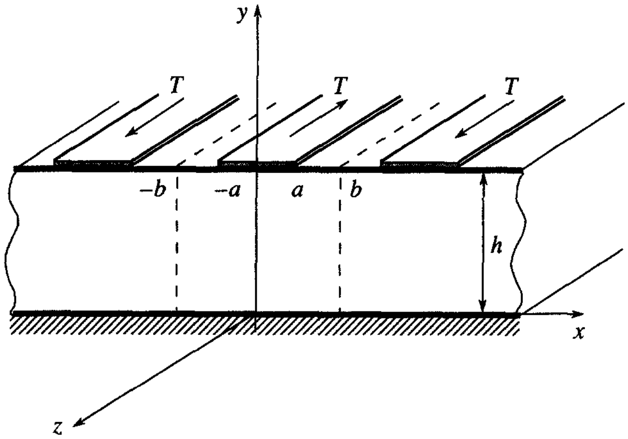

Fig. 1

Substituting expressions (1.3) and (1.4) into the equilibrium equation, we obtain the Lamé equation

$$\partial^2 w/\partial x^2 + 2\kappa\partial w/\partial y + \partial^2 w/\partial y^2 = 0 \tag{1.5}$$

where the quantity κ occurs in the exponential term of formula (1.1).

In the case of the primary punch, the boundary conditions on the edges of the layer are the same for both problems and have the form

$$y = 0: w = 0 \tag{1.6}$$

$$y = h: w = \varepsilon, \quad (|x| \le a), \quad \tau_{yz} = 0 \to \partial w/\partial y = 0 \quad (a < |x| \le b) \tag{1.7}$$

and, when $x = \pm b$, the boundary conditions are different and have the form

$$w = 0 \text{ for problem 1}, \; \tau_{xz} = 0 \to \partial w/\partial x = 0 \text{ for problem 2} \tag{1.8}$$

2. REDUCTION OF THE PROBLEMS TO AN INTEGRAL EQUATION

We will now consider the following subsidiary Problems 1a and 2a, which differ from the corresponding Problems 1 and 2 in that the boundary conditions (1.6) and (1.8) are retained while conditions (1.7) become

$$y = h: \tau_{yz} = G(y)\frac{\partial w}{\partial y} = \tilde{\tau}(x); \quad \tilde{\tau}(x) = \begin{cases} \tau(x), & |x| \le a \\ 0, & a < |x| \le b \end{cases} \tag{2.1}$$

For Problems 1a and 2a, on expanding the functions $w(x, y)$ and $\tilde{\tau}(x)$ in Fourier series in the interval $-b \le x \le b$

$$w(x, y) = \sum_{k=0}^{\infty} w_k(y)\cos\beta_k x, \quad \tilde{\tau}(x) = \sum_{k=0}^{\infty} \tau_k \cos\beta_k x$$

$$\tau_0 = \frac{1}{2b}\int_{-a}^{a} \tau(\xi)d\xi, \quad \tau_k = \frac{1}{b}\int_{-a}^{a} \tau(\xi)\cos\beta_k\xi d\xi$$

$$\beta_k = \pi u_k/b; \; u_k = k - 1/2 \quad \text{for problem 1a}, \; u_k = k \quad \text{for problem 2a}; \quad k = 1, 2, \dots \tag{2.2}$$

(for Problem 1a, $w_0(y) \equiv 0$, $\tau_0 = 0$) and using the method of separation of variables, we solve the subsidiary Problems 1a and 2a for Eq. (1.5). As a result, we obtain

$$w(x,y) = \frac{1}{G(h)}\sum_{k=0}^{\infty}\tau_k W_k(y)\cos(\beta_k x)$$
$$W_k(y) = \frac{\operatorname{sh}\chi_k y}{-\kappa\operatorname{sh}\chi_k h + \chi_k \operatorname{ch}\chi_k h}, \quad \chi_k = \sqrt{\kappa^2+\beta_k^2} \tag{2.3}$$

Substituting the coefficients τ_k in the form of (2.2) into expression (2.3) and putting $y = h$, after some reduction we find

$$w(x,h) = \frac{1}{2bG(h)}\int_{-a}^{a}\tau(\xi)\sum_{k=-\infty}^{\infty}W_k(h)\exp[i\beta_k(\xi-x)]d\xi \tag{2.4}$$

Note that, when solving the subsidiary problems, all the boundary conditions of the basic Problems 1 and 2 were satisfied (apart from the first condition of (1.7)). On now satisfying the remaining boundary condition using formula (2.4), we arrive at the following integral equation of the first kind with a difference kernel in the unknown function for the distribution of the shear contact forces $\tau(x)$

$$\int_{-a}^{a}\tau(\xi)K\left[\frac{\pi}{b}(\xi-x)\right]d\xi = \pi G(h)\varepsilon \quad (|x|\le a) \tag{2.5}$$

$$K(s) = \frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{L(\beta u_k)}{u_k}\exp(iu_k s), \quad \beta = \frac{\pi h}{b} \tag{2.6}$$

$$L(\upsilon) = \frac{\upsilon\operatorname{sh}\sqrt{m^2+\upsilon^2}}{-m\operatorname{sh}\sqrt{m^2+\upsilon^2}+\sqrt{m^2+\upsilon^2}\operatorname{ch}(m^2+\upsilon^2)}, \quad m = \kappa h \tag{2.7}$$

The kernel (2.6) is a periodic function, and, moreover, it can be shown that it is irregular: when $s \to 0$, it behaves as $-\ln|s|$.

3. REDUCTION OF THE INTEGRAL EQUATION (2.5) TO A SINGULAR INTEGRAL EQUATION

In Eq. (2.5), we will change to dimensionless variables and dimensionless quantities using the formulae

$$\xi' = \frac{\xi}{a}, \quad x' = \frac{x}{a}, \quad \varphi(x') = \frac{\tau(ax')}{G(h)}, \quad f = \frac{\varepsilon}{a}, \quad \alpha = \frac{\pi a}{b} \tag{3.1}$$

As a result, we obtain

$$\int_{-1}^{1}\varphi(\xi)K[\alpha(\xi-x)]d\xi = \pi f \quad (|x|\le 1) \tag{3.2}$$

(we shall henceforth omit the primes) Note that, by virtue of formulae (2.6) and (2.7), the integral equation (3.2) contains three dimensionless parameters α, β and m where $0 < \alpha < \pi$, $0 < \beta < \infty$, $0 \le |m| < \infty$.

With regard to the function $L(\upsilon)$, defined by formula (2.7), we can conclude that it is odd, continuous and does not vanish for all υ, $0 < |\upsilon| < \infty$. Moreover, the following asymptotic relations hold for it

$$L(\upsilon) = 1 + O(\upsilon^{-1}) \ (|\upsilon|\to\infty), \quad L(\upsilon) = A\upsilon + O(\upsilon^{-3}) \ (\upsilon\to 0)$$
$$A = \operatorname{sh} m e^m m^{-1} \tag{3.3}$$

By virtue of the properties (3.3) of the function $L(\upsilon)$, it can be represented in the form

$$L(\upsilon) = \operatorname{th} A\upsilon + g(\upsilon)$$
$$g(\upsilon) = O(\upsilon^{-1}) \ (|\upsilon| \to \infty), \quad g(\upsilon) = O(\upsilon^3) \ (\upsilon \to 0) \tag{3.4}$$

and, on the half-line $|\upsilon| \in (0, \infty)$, the function $L(\upsilon)$ has a single extremum, a maximum.

We now consider the series

$$M(s) = \sum_{k=1}^{\infty} \operatorname{th}\gamma u_k \sin u_k s, \quad \gamma = \beta A \tag{3.5}$$

For Problems 1 and 2, we have ([4, formulae 1.441(2), 1.442(2) and 8.146 (10, 12)])

$$M_1(s) = \frac{1}{2}\left[\operatorname{cosec}\frac{s}{2} - \sum_{k=1}^{\infty} \frac{q^{2k-1}}{1+q^{2k-1}} \sin\left(k - \frac{1}{2}\right)s\right] = \mathrm{K}(e)F_1(u)$$

$$M_2(s) = \frac{1}{2}\left[\operatorname{ctg}\frac{s}{2} - 4\sum_{k=1}^{\infty} \frac{q^{2k}}{1+q^{2k}} \sin ks\right] = \mathrm{K}(e)F_2(u); \quad q = e^{-\gamma} \tag{3.6}$$

$$F_1(u) = \frac{\operatorname{dn} u}{\operatorname{sn} u}, \quad F_2(u) = \frac{\operatorname{cn} u}{\operatorname{sn} u}; \quad u = \frac{\mathrm{K}(e)s}{\pi}$$

where $M_1(s)$ corresponds to Problem 1 and $M_2(s)$ to Problem 2. The quantity $e < 1$ is determined from the transcendental equation

$$\pi\mathrm{K}(\sqrt{1-e^2})/\mathrm{K}(e) = \gamma \tag{3.7}$$

where $\mathrm{K}(e)$ is a complete elliptic integral of the first kind, and $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ are Jacobian elliptic functions.

We now differentiate integral equation (3.2) once with respect to x and, on the basis of relations (3.4)–(3.6), we write it in the form

$$\mu \int_{-1}^{1} \varphi(\xi) F[\mu(\xi - x)] d\xi = -\alpha \int_{-1}^{1} \varphi(\xi) G_*[\alpha(\xi - x)] d\xi$$
$$\mu = \pi^{-1}\mathrm{K}(e)\alpha, \quad G_*(s) = \sum_{k=1}^{\infty} g(\beta u_k) \sin u_k s \tag{3.8}$$

The function $F(u)$ is equal to $F_1(u)$ or $F_2(u)$ and, on the basis of the properties (3.4) of the function $g(\upsilon)$, it can be shown that the function $G_*(s)$ is bounded when $s \le 2\alpha$. A more detailed investigation leads to the conclusion that the function $G_*(s)$ is continuous when $s \neq 0$ and that there is a discontinuity in the neighbourhood of $s = 0$, that is, it behaves as follows:

$$G_*(s) = \frac{\pi - s}{2} \operatorname{sgn} s \tag{3.9}$$

It is natural to complete the definition of the function $G_*(s)$ putting $G_*(0) = 0$.

Note that

$$F_1[\mu(\xi - x)] - F_1[\mu(\xi + x)] = 2\operatorname{sn}\mu x \operatorname{cn}\mu\xi \operatorname{dn}\mu x/\Delta$$

$$F_2[\mu(\xi - x)] - F_2[\mu(\xi + x)] = 2\operatorname{sn}\mu x \operatorname{cn}\mu x \operatorname{dn}\mu\xi/\Delta$$

$$\Delta = \operatorname{sn}^2\mu\xi - \operatorname{sn}^2\mu x$$

On the basis of these equalities and taking account of the fact that $\mu < \mathrm{K}(e)$ and the functions $\mathrm{cn}(\mathrm{K}(e)x)$ and $\mathrm{dn}(\mathrm{K}(e)x)$ decrease monotonically from 1 to 0 as x increases from 0 to 1 [5], we reduce Eq. (3.8) to the form:
for problem 1

$$\mu\int_{-1}^{1}\frac{\varphi(\xi)\mathrm{cn}\mu\xi}{\mathrm{sn}\mu\xi-\mathrm{sn}\mu x}d\xi = -\frac{\alpha}{\mathrm{dn}\mu x}\int_{-1}^{1}\varphi(\xi)G_*[\alpha(\xi-x)]d\xi \tag{3.10}$$

and for problem 2,

$$\mu\int_{-1}^{1}\frac{\varphi(\xi)\mathrm{dn}\mu\xi}{\mathrm{sn}\mu\xi-\mathrm{sn}\mu x}d\xi = -\frac{\alpha}{\mathrm{cn}\mu x}\int_{-1}^{1}\varphi(\xi)G_*[\alpha(\xi-x)]d\xi \tag{3.11}$$

Taking account once again of the fact that $\mu < \mathrm{K}(e)$ and that the function $\mathrm{sn}(\mathrm{K}(e)x)$ increases monotonically from 0 to 1 as x increases from 0 to 1 [5], we introduce the new variables and notation

$$\tau = \mathrm{sn}\mu\xi, \quad t = \mathrm{sn}\mu x, \quad c = \mathrm{sn}\mu \tag{3.12}$$

and yet another function which is inverse to $\mathrm{sn}\, u$

$$\xi = \frac{\mathrm{asn}\tau}{\mu}, \quad x = \frac{\mathrm{asn}t}{\mu}, \quad \mathrm{asn}t = \int_0^t\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}} \tag{3.13}$$

(the definition of the function $\mathrm{sn}\, u$ given in [4, formula 8.144(1)] has been used).
On the basis of relations (3.12) and (3.13), we reduce Eqs (3.10) and (3.11) to the form

$$\int_{-c}^{c}\frac{\psi_j(\tau)}{\tau-t}d\tau = -\int_{-c}^{c}\psi_j(\tau)H_j(\tau,t) \quad (|t|\le c) \tag{3.14}$$

where $j = 1$ corresponds to Problem 1, $j = 2$ corresponds to Problem 2 and the notation

$$\frac{\varphi(\xi)}{\mathrm{dn}\mu\xi} = \psi_1(\tau), \quad \frac{\varphi(\xi)}{\mathrm{cn}\mu\xi} = \psi_2(\tau)$$

$$H_j(\tau,t) = \frac{\pi\tilde{H}_j(\tau,t)}{\mathrm{K}(e)}G_*\left[\frac{\pi}{\mathrm{K}(e)}(\mathrm{asn}\tau-\mathrm{asn}t)\right] \tag{3.15}$$

$$\tilde{H}_1(\tau,t) = \frac{1}{\sqrt{1-e^2t^2}\sqrt{1-\tau^2}}, \quad \tilde{H}_2(\tau,t) = \tilde{H}_1(t,\tau)$$

has been introduced.
It is important to note that $c < 1 < 1/e$ and that the root singularities in the denominators of the expressions $\tilde{H}_j(\tau, t)$ lie outside the intervals of definition and integration in (3.14). Hence, $H_j(\tau, t)$ are the bounded parts of the kernels of the singular integral equations (3.14) (a singular integral operator with a Cauchy kernel occurs on the left-hand side of (3.14)).

4. APPROXIMATE SOLUTION OF THE SINGULAR INTEGRAL EQUATION (3.14)

In this section, we will omit the subscript j, since the schemes for the approximate solution of Problems 1 and 2 are the same.
Taking account of the properties of the function $G_*(s)$, it can be proved [6] that, if a solution of integral equation (3.14) exists for fixed values of the parameters α, β and m in a class of functions for which the integral

$$\int_{-c}^{c} |\psi(\tau)|^p d\tau \quad (0 < p < 2) \tag{4.1}$$

converges, then this solution has the structure

$$\psi(t) = \Psi(t)(c^2 - t^2)^{-1/2} \tag{4.2}$$

and the function $\Psi(t)$ is bounded at least when $|t| \le c$.

Note that the function $\psi(t)$ is found from the singular integral equation (3.14) apart from the term

$$\pi^{-1} C(c^2 - t^2)^{-1/2} \tag{4.3}$$

The constant C is then determined from the supplementary condition, which will be discussed in Section 5.

Next, we use the well-known Multhopp–Kalandiya method [6–8], which we shall briefly describe as it applies to the problem being considered.

We substitute expression (4.2) into Eq. (3.14) and change to the new variables ω and θ according to the formulae

$$\tau = c\cos\omega, \quad t = c\cos\theta \tag{4.4}$$

As a result, we obtain

$$\begin{gathered}\frac{1}{c}\int_0^\pi \frac{\Omega(\omega)d\omega}{\cos\omega - \cos\theta} = -\int_0^\pi \Omega(\omega)H(c\cos\omega, c\cos\theta)d\omega, \quad 0 \le \theta \le \pi \\ \Omega(\theta) = \Psi(c\cos\theta)\end{gathered} \tag{4.5}$$

We now construct a Lagrangian interpolation polynomial for the function $\Psi(t)$ with respect to the nodes

$$t_n = c\cos\theta_n, \quad \theta_n = \pi(2n-1)/(2N), \quad n = 1, 2, \dots N \tag{4.6}$$

which are the zeros of a Chebyshev polynomial of the first kind $T_N(t/c)$ [4]. In the special case when $N = 2r + 1$ $(r \ge 1)$, this polynomial has the form [8]

$$\Omega(\theta) \approx \frac{1}{r + 1/2}\sum_{n=1}^{r+1} \Omega(\theta_n)\delta_n\left(1 + 2\sum_{l=1}^{r} \cos 2l\theta_n \cos 2l\theta\right), \quad \delta_n = \begin{cases}1, & n \ne r+1 \\ 1/2, & n = r+1\end{cases} \tag{4.7}$$

Substituting the approximate expression (4.7) for $\Omega(\theta)$ into the integral equation (4.5) and making use of the relation [4, formula 7.344 (1)]

$$\int_0^\pi \frac{\cos l\omega}{\cos\omega - \cos\theta} d\omega = \pi\frac{\sin l\theta}{\sin\theta}, \quad 0 \le \theta \le \pi, \quad l = 0, 1, \dots \tag{4.8}$$

we evaluate the integral on the left-hand side of Eq. (4.5) exactly. For the approximate evaluation of the integral on the right-hand side of this equation, we use the Gaussian quadrature formula [7, 8]

$$\int_0^\pi f(\omega)d\omega = \frac{\pi}{N}\sum_{n=1}^{N} f(\theta_n) \tag{4.9}$$

After evaluating the integrals in (4.5), we put $\theta = \theta_s$ in the resulting relation and arrive at a system of r homogeneous linear algebraic equations in $\Omega(\theta_n)$

$$\sum_{n=1}^{r+1} \Omega(\theta_n)\delta_n \left\{ \frac{1}{c\sin\theta_s}\chi_r(\theta_n, \theta_s) + \frac{1}{2}[H(c\cos\theta_n, c\cos\theta_s) + \right.$$

$$\left. + H(-c\cos\theta_n, c\cos\theta_s)] \right\} = 0, \quad s = 1, 2, \dots r \tag{4.10}$$

$$\chi_r(\omega, \theta) = 2\sum_{m=1}^{r} \cos 2l\omega \sin 2l\theta$$

5. SUPPLEMENTARY CONDITION

In system (4.10), there are r equations but $r + 1$ unknowns $\Omega(\theta_n)$. In order to close the system, it is necessary to obtain a further inhomogeneous equation. This can be done in the following way.

We recall that Eq. (3.8), on the basis of which system (4.10) was obtained, differs by a single operation of differentiation from Eq: (3.2). Consequently, the algebraic equation in the quantities $\Omega(\theta_n)$, which is required, must be obtained from Eq: (3.2). At the same time, here it is possible to put x equal to any value: it is convenient to put $x = 0$.

So, from Eq: (3.2) when $x = 0$, taking relation (3.4) into account, we have

$$\int_{-1}^{1} \varphi(\xi)P_j(\alpha\xi)d\xi = \pi f - \int_{-1}^{1} \varphi(\xi)Q_j(\alpha\xi)d\xi$$

$$P_1(\alpha\xi) = \sum_{k=1}^{\infty} \frac{\operatorname{th}\gamma(k-1/2)}{k-1/2}\cos\left(k-\frac{1}{2}\right)\alpha\xi = \frac{1}{2}\ln\frac{1+\operatorname{cn}\mu\xi}{1-\operatorname{cn}\mu\xi}$$

$$P_2(\alpha\xi) = \frac{1}{2}\gamma + \sum_{k=1}^{\infty} \frac{\operatorname{th}\gamma k}{k}\cos k\alpha\xi = \frac{1}{2}\ln\frac{1+\operatorname{dn}\mu\xi}{1-\operatorname{dn}\mu\xi} \tag{5.1}$$

$$Q_j(\alpha\xi) = \sum_{k=1}^{\infty} \frac{g(\beta u_k)}{u_k}\cos u_k\alpha\xi$$

Here, the series for $P_1(\alpha\xi)$ and $P_2(\alpha\xi)$ are summed using the formulae from [1, 2]. The functions $Q_j(\alpha\xi)$ are continuous when $|\xi| \le 1$, which can be proved on the basis of properties (3.4) of the functions $g(\upsilon)$.

Making substitutions in (5.1) using formulae (3.12), (3.13) and (3.15), after some reduction we obtain

$$-\int_{-c}^{c} \psi_j(\tau)L_j(\tau)d\tau = \pi\mu f - \int_{-c}^{c} \psi_j(\tau)N_j(\tau)d\tau$$

$$N_j(\tau) = R_j(\tau)\left[S_j(\tau) + Q_j\left(\frac{\pi a \operatorname{sn}\tau}{\mathrm{K}(e)}\right)\right] \tag{5.2}$$

$$L_1(\tau) = \ln|\tau|, \quad L_2(\tau) = L_1(e\tau); \quad R_1(\tau) = (1-\tau^2)^{-1/2}, \quad R_2(\tau) = R_1(e\tau)$$

$$S_1(\tau) = (\sqrt{1-\tau^2}-1)\ln|\tau| + \ln(\sqrt{1-\tau^2}+1), \quad S_2(\tau) = S_1(e\tau)$$

Now, substituting expression (4.2) into (5.2) and changing to the new variables and notation as given by formulae (4.4) and (4.5), we rewrite equality (5.2) in the form

$$-\int_{0}^{\pi} \Omega_j(\omega)L_j(c\cos\omega)d\omega = \pi\mu f - \int_{0}^{\pi} \Omega_j(\omega)N_j(c\cos\omega)d\omega \tag{5.3}$$

Table 1

α	Problem 1			Problem 2		
	β = 2	4	8	β = 2	4	8
			$m = -1$			
π/9	1.870	1.408	1.194	1.866	1.342	0.942
2π/9	2.738	1.933	1.585	2.728	1.804	1.164
π/3	3.567	2.419	1.947	3.543	2.202	1.334
4π/9	4.394	2.912	2.319	4.344	2.567	1.477
			$m = 1$			
π/9	0.656	0.720	0.838	0.560	0.371	0.218
2π/9	0.812	0.888	1.051	0.663	0.408	0.230
π/3	0.963	1.049	1.252	0.750	0.434	0.237
4π/9	1.131	1.228	1.470	0.826	0.455	0.242

Using the relation [3, 9]

$$-\int_0^{\pi} \cos 2l\omega \ln|c\cos\omega|\, d\omega = \begin{cases} \pi\ln(2/c), & l = 0 \\ \pi(-1)^l/(2l), & l \neq 0 \end{cases} \tag{5.4}$$

we substitute expression (4.7) into the left-hand side of relation (5.3) and evaluate the integral on the left-hand side of (5.3) exactly. For the approximate evaluation of the integral on the right-hand side, we again make use of the quadrature formula (4.9). As a result, we find a further equation in $\Omega(\theta_n)$

$$\sum_{n=1}^{r+1} \Omega_j(\theta_n)\delta_n \left[l_j\left(\frac{2}{c}\right) + \sum_{l=1}^{r} (-1)^l \frac{\cos 2l\theta_n}{l} + N_j(c\cos\theta_n) \right] = (r + 1/2)\mu f$$
$$l_1\left(\frac{2}{c}\right) = \ln\frac{2}{c}, \quad l_2\left(\frac{2}{c}\right) = l_1\left(\frac{2}{ec}\right) \tag{5.5}$$

which supplements system (4.10).

6. DETERMINATION OF THE RELATION BETWEEN THE SHEAR FORCE AND THE MAGNITUDE OF THE DISPLACEMENT OF THE PUNCH

Note that each punch must be in equilibrium under the action of the shear force T on its upper edge and the contact shear stress on its lower edge. This equilibrium condition for the primary punch has the form

$$T = \int_{-a}^{a} \tau(\xi)\, d\xi \tag{6.1}$$

In the integrand of formula (6.1), we now make the following transition

$$\tau(\xi) \to \varphi(\xi') \to \psi(\tau) \to \Psi(\tau) \to \Omega(\omega)$$

and then substitute $\Omega(\omega)$ in the form of (4.7) into it. As a result, we obtain

$$\frac{T}{G(h)a} = \frac{2}{\mu(r+1/2)} \sum_{n=1}^{r+1} \Omega_j(\theta_n)\delta_n \left[J_0^{(j)}(c) + 2\sum_{l=1}^{r} (-1)^l \cos 2l\theta_n J_{2l}^{(j)}(c) \right]$$
$$J_{2l}^{(1)}(c) = \int_0^{\pi/2} \frac{\cos 2l\omega}{\sqrt{1 - c^2\sin^2\omega}}\, d\omega, \quad J_{2l}^{(2)}(c) = J_{2l}^{(1)}(ce), \quad l = 0, 1, \ldots \tag{6.2}$$

All the integrals $J_{2l}^{(1)}(c)$ can be expressed in terms of complete elliptic integrals of the first kind $\mathrm{K}(c)$ and the second kind $\mathrm{E}(c)$. The formulae for the first four integrals are

$$J_0^{(1)}(c) = \mathrm{K}(c), \quad J_2^{(1)}(c) = [(-2+c^2)\mathrm{K}(c)+2\mathrm{E}(c)]c^{-2}$$

$$J_4^{(1)}(c) = [(16-16c^2+3c^4)\mathrm{K}(c)+(-16+8c^2)\mathrm{E}(c)](3c^4)^{-1}$$

$$J_6^{(1)}(c) = [(-256+384c^2-158c^4+15c^6)\mathrm{K}(c)+$$
$$+(256-256c^2+46c^4)\mathrm{E}(c)](15c^4)^{-1}$$

After solving system (4.10), (5.5) for $\Omega(\theta_n)$, an approximate relation between the dimensionless shear force $N = T/(G(h)a)$ and the dimensionless magnitude of the displacement of the punch $f = \varepsilon/a$ can be found using formula (6.2).

Values of N/f for Problems 1 and 2 when $m = -1$ and $m = 1$ and for different values of α and β are presented in Table 1.

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